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Three Dimensional Large Amplitude Shallow Water Wave

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Abstract

We numerically calculated solutions of the full non-linear fundamental water wave equation to study the behavior of three dimensional periodic water waves in shallow water. In shallow water, unlike deep-water waves, the dispersiveness can be balanced with the non-linearity and waves can form solitary waves. Korteweg-de Vries (KdV) equation has been well known as a model describes such solitary wave motions of weakly non- linear propagating in one direction. A still more interesting problem should be interactions of solitary waves. Those phenomena with weakly non-linear can be approximated by Kadomtsev Petviahvili (KP) equation that is an extended model of KdV equation. We focus on even stronger non-linearity wave motions than weakly non-linearity for which KP will be valid. The behavior of harmonic resonance in a periodic solution in this study is a part of our interest.

1. Introduction

In this study, we investigate interactions of two solitary waves in shallow water. Particularly, we are interested in interactions of large amplitude solitary waves and properties of periodic solutions of these. Thus we extend cases of weakly nonlinear interactions to cases of strong nonlinear interactions by using numerical schemes, such as the Newton method and the Galerkin method and obtain periodic steady state solutions from fundamental equations for water waves. As a property of periodic solutions, obtained numerical solutions deviate from Miles' theoretical values in some conditions [1, 2]. Miles' theory is based on the third order approximation and describes asymptotic solutions $t \to \infty$. Harmonic resonances are suspected causes of these deviations but the causes are not fixed because of the complexity of nonlinearity. In comparison between interactions of weak nonlinear cases and these of strong nonlinear cases, there are some differences in the length of stems and the ratio $\alpha = a_M/a_i$ (a_M is the maximum wave height divided by the uniform depth d. a_i is an incident solitary wave height divided by the uniform depth d) caused by the limitation of solitary wave heights.

2. Formulation of the problem

2.1 Fundamental equation for water waves

We consider a gravity wave on an inviscid, incompressible fluid of uniform depth and also irrotational flow is assumed. d is the uniform depth, ϕ is the velocity potential, x and y are horizontal coordinates, z is the vertical coordinate, $z = \eta(x,y,t)$ is the surface displacement and g is the gravitational acceleration. Fundamental equations for water waves are written as

$$\Delta\phi = 0 \text{ for } z \le \eta(x,y,t), \tag{2.1}$$

$$\frac{P}{\rho} = \phi_t + \frac{1}{2}\nabla\phi\cdot\nabla\phi + gz = 0 \text{ on } z = \eta(x,y,t), \tag{2.2}$$

$$\frac{D}{Dt}\left(\frac{P}{\rho}\right) = \left(\frac{\partial}{\partial t} + \nabla\phi\cdot\nabla\right)\left[\phi_t + \frac{1}{2}\nabla\phi\cdot\nabla\phi + gz\right] = 0 \text{ on } z = \eta(x,y,t), \tag{2.3}$$

$$\phi_z = 0 \text{ on } z = -d. \tag{2.4}$$

2.2 Formulation for numerical calculation

In order to calculate a steady progressive wave, we consider moving coordinate and we normalize the variables as follows.

$$\begin{gathered}(x^*, y^*, z^*, H^*) = (Kx, Ky, Kz, KH),\\ t^* = \omega t,\ \Phi^* = \frac{K^2}{\omega}\phi,\\ T = px^* - t^*,\ Y = qy^*,\ Z = z^*,\end{gathered} \tag{2.5}$$

where K is a wave number, ω is a frequency of an incident water wave and $p = \sin\theta_i$ and $q = \cos\theta_i$. Here, θ_i denotes the initial angle of the Fourier mode. When a wave number vector of the Fourier mode is (k_x, k_y), we have relations as

$$k_x = K\sin\theta_i,\ k_y = K\cos\theta_i,\ \tan\theta_i = k_x/k_y. \tag{2.6}$$

Then

$$\begin{gathered}\Phi(Y,Z,T) = \Phi^*(x^*, y^*, z^*, H^*),\\ H(Y,T) = H(x^*, y^*, t^*).\end{gathered} \tag{2.7}$$

Then the form of the fundamental equations becomes

$$p^2\Phi_{TT}+q^2\Phi_{YY}+\Phi_{ZZ}=0 \text{ for } Z\leq H(Y,T), \tag{2.8}$$

$$P(Y,Z,T)=-\Phi_T+GZ +\frac{1}{2}\left(p^2{\Phi_T}^2+q^2{\Phi_Y}^2+{\Phi_Z}^2\right)=0 \quad \text{on } Z=H(Y,T), \tag{2.9}$$

$$\begin{aligned}Q(Y,Z,T)&=\Phi_{TT}\\&+p^2\Phi_T(-2\Phi_{TT}+p^2\Phi_T\Phi_{TT}+q^2\Phi_Y\Phi_{YT}+\Phi_Z\Phi_{ZT})\\&+q^2\Phi_Y(-2\Phi_{YT}+p^2\Phi_T\Phi_{YT}+q^2\Phi_Y\Phi_{YY}+\Phi_Z\Phi_{YZ})\\&+\Phi_Z(-2\Phi_{ZT}+p^2\Phi_T\Phi_{ZT}+q^2\Phi_Y\Phi_{YZ}+\Phi_Z\Phi_{ZZ}+G)\\&=0 \text{ on } Z=H(Y,T),\end{aligned} \tag{2.10}$$

$$\Phi_Z=0 \text{ on } Z=-d. \tag{2.11}$$

We define the maximum wave height as

$$a_M=[H(0,0)-H(\pi,0)]. \tag{2.12}$$

Assuming the velocity potential Φ as periodic, we have a truncated series

$$\begin{aligned}\Phi(Y,Z,T)=\sum_{k=0}^{N}\sum_{j=1}^{N}A_{kj}[\cosh(\alpha_{kj}Z)\\+\sinh(\alpha_{kj}Z)\tanh(\alpha_{kj}d)]\cos(kY)\sin(jT),\\ \alpha_{kj}=\sqrt{p^2j^2+q^2k^2}.\end{aligned} \tag{2.13}$$

3. Numerical scheme

Applying Newton's method to (2.9), the recurrent formula for Newton's method to calculate the surface displacement H is

$$H_{n+1}=H_n-\frac{dZ}{dP(Y,H_n,T)}P(Y,H_n,T). \tag{3.1}$$

Then using Galerkin's method to obtain the independent relations for unknowns A_{kj}, we have

$$\begin{aligned}F_{lm}(A_{kj},G)\\=\int_0^{\pi}dY\int_0^{\pi}dTQ(Y,H,T)\cos(lY)\sin(mT)\\=0.\end{aligned} \tag{3.2}$$

Because when $l+m$ is odd, (3.2) is trivial, the number of independent relations (3.2) is $N(N+1)/2$. Another independent relation is expressed as

$$W(A_{kj},G)=a_M-[H(0,0;A_{kj},G)-H(\pi,0;A_{kj},G)]=0. \tag{3.3}$$

Finally, we can obtain the sufficient number of independent relations and we can solve the nonlinear equations (3.2) and (3.3). Here, a result of the third order approximation for short-crested wave calculated by Hsu *et. al.* [3] is used as the initial solution of iterations.

4. Result

4.1 Weakly nonlinear cases

We discuss weakly nonlinear interactions; $0.02<a_i<0.04$. Miles' theory is based on the third order approximation. Because of our symmetric assumption of a solution, any solution cannot be found for the interaction parameter $\kappa<1$ where $\kappa=\cos\psi/(\tan\psi\sqrt{3a_i})$ [4] and ψ is the angle between obtained incident wave crest and Y-axis (see Fig 4.3). Most of our results are consistent with those obtained from Miles' theory as shown in Fig 4.1. However, some large or small deviations exist in scattered state. Apparently, the harmonic resonance is a cause of these deviations and we will investigate this in the next section in more detail. Stems suddenly extend in Y direction as κ approaches $\kappa=1.0$ from $\kappa=1.1$, which is called a soliton resonance [2].(see Fig.4.2)

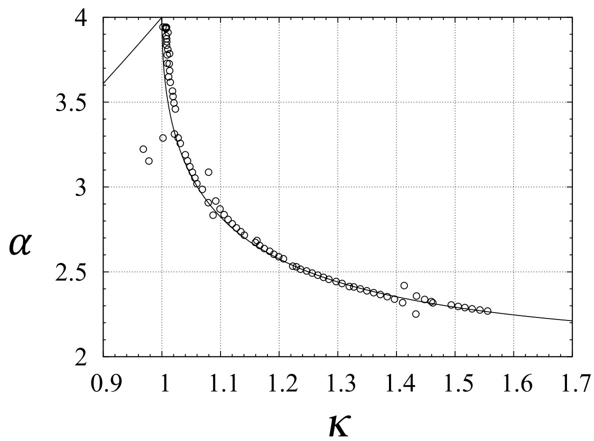

Figure 4.1. The ratio $\alpha=a_M/a_i$ versus κ for $d=0.050$. —— : Miles' theory; ○: the present result.

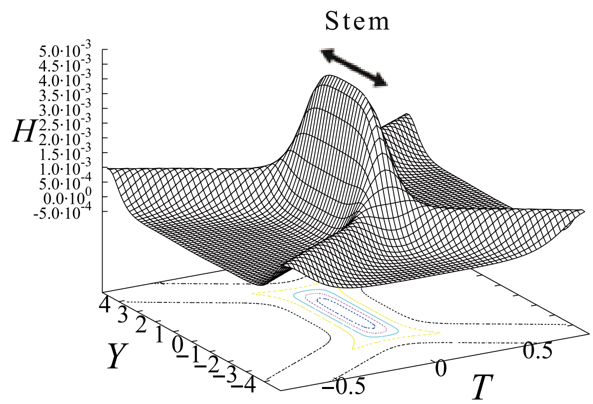

Figure 4.2. The wave profile for $\kappa=1.0$ and $d=0.050$.

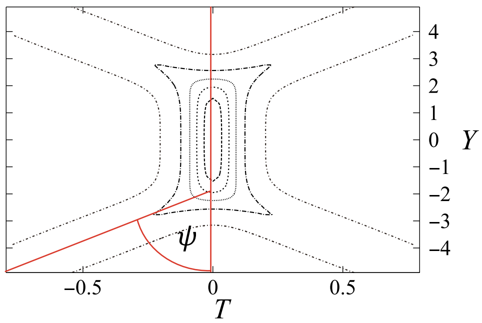

Figure 4.3. The contour for $\kappa = 1.0$ and $d = 0.050$.

4.2 Harmonic resonance

We investigate the deviations where depths $d = 0.050$ and $d = 0.090$. Fig 4.4 shows the chosen region denoted by squares (a) and (b) for the depth $d = 0.050$ and the enlargement of the region (a).

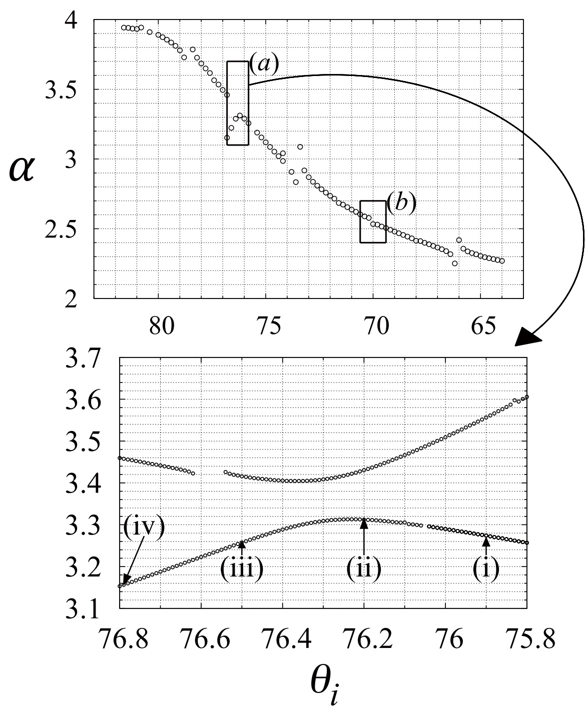

Figure 4.4. The ratio $\alpha = a_M/a_i$ versus the angle of the Fourier modes θ_i for $d = 0.050$.

Fig 4.5 shows the value of the coefficient difference $|\Delta A_{kj,\theta_i}|$ defined as (4.1).

$$\Delta A_{kj,\theta_i} = A_{kj,\theta_i+\delta\theta_i} - A_{kj,\theta_i} \quad (4.1)$$

We can see that certain coefficients become considerably large such as A_{64} and A_{75} in Fig 4.5, which deviate our results from Miles' theory. Harmonic resonance of a water wave for a finite depth is known to exist [5] if an interaction of waves satisfies

$$\alpha_{kj}\tanh(\alpha_{kj}d) = j^2\tanh(d). \quad (4.2)$$

However, we have not been able to find good agreements between a harmonic resonance angle θ_{HR} and an angle of our result where the deviation occurs.

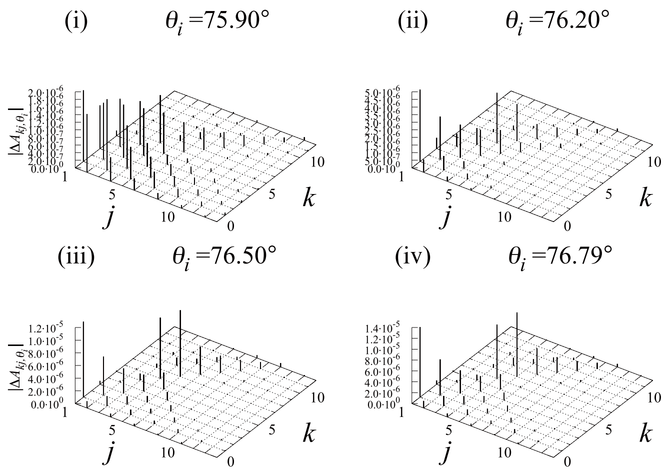

Figure 4.5. The value of the coefficient difference $|\Delta A_{kj,\theta_i}|$.

4.3 Strong nonlinear cases

In strong nonlinear cases, our numerical result does not agree with Miles' theoretical value, which is an inevitable result because the nonlinearity parameter a_i is out of Miles' approximation of the weakly nonlinearity $a_i \ll 1$. (see Fig. 46) Bad convergences frequently occur for some angles θ_i and result in rough wave profiles. We consider that these phenomena are the same phenomena as that we have found in weakly non-linear cases.

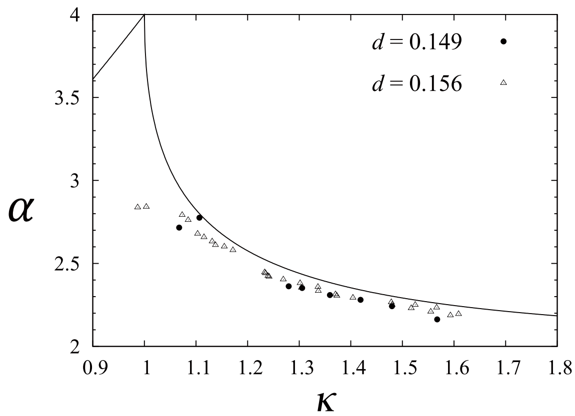

Figure 4.6. The ratio $\alpha = a_M/a_i$ versus κ. —— : Miles' theory; ○: the present result.

Differences between present results and Miles' theoretical value increase as κ approaches $\kappa = 1.0$. A Mach stem made by an interaction of two solitary waves becomes a steady solitary wave and the maximum solitary wave height is known as 0.827 [6]. We consider that even if an incident wave height a_i

increases, the maximum height a_M is suppressed within the maximum solitary wave height of 0.827. This is why the ratio α of the maximum height a_M to an incident wave height a_i decreases as an incident wave height a_i increases.

Next, we examine wave profiles for $\kappa = 1.1$ and $\kappa = 1.0$ with incident wave heights $a_i = 0.1$, $a_i = 0.2$ and $a_i = 0.3$ (see the wave profile and the contour in Fig. 4.7 and Fig. 4.8 for $a_i = 0.3$). In comparison with a weakly nonlinear case, when $\kappa = 1.1$, there is little difference between a weakly and a strong nonlinear cases. However, when $\kappa = 1.0$, there is a clear difference between them in the length of a stem. The length of a stem in a strong nonlinear case is shorter than that in a weakly nonlinear case. And this difference becomes more remarkable as an incident wave height a_i increases.

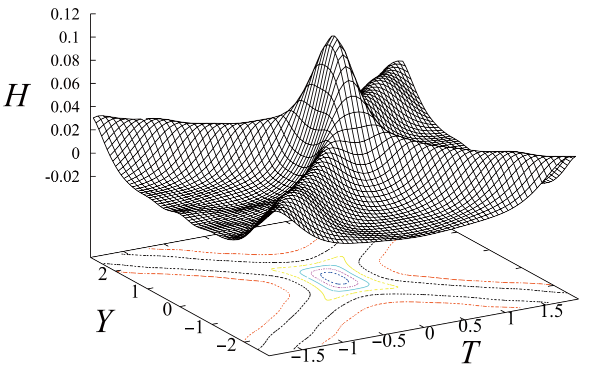

Figure 4.7. The wave profile for $\kappa = 1.0$ and $a_i = 0.3$.

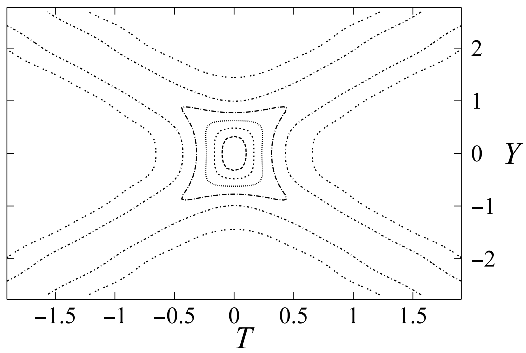

Figure 4.8. The contour for $\kappa = 1.0$ and $a_i = 0.3$.

5. Conclusions

The scheme used in this research was successful to obtain a solution for three-dimensional large amplitude shallow water wave. When incident wave heights a_i is small, most of our numerical results show good agreement with Miles' theoretical values. However, some deviations exist and we observed such wave profiles were rough and their convergences were worse than others. We found that as an incident wave height a_i increases, the ratio α of the maximum wave height a_M to an incident wave height a_i started decreasing and accordingly, a difference between the present result and Miles' theory increased. We investigated the cause of those deviations. We found that certain coefficients A_{kj} for wave numbers (k, j) became considerably large and those wave numbers (k, j) depend on the incident angle θ_i. We compared our result with a harmonic resonance angle θ_{HR}. However, we could not find good agreement in comparisons between our numerical result and a harmonic resonance angle θ_{HR}.When $\kappa = 1.0$, since the maximum height a_M is suppressed within the maximum solitary wave height of 0.827, the length of a stem in a strong nonlinear case was shorter than that in a weakly nonlinear case.

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